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Argyres-Seiberg duality and the Higgs branch

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Abstract: We demonstrate the agreement between the Higgs branches of two $\mathcal{N} = 2$ theories proposed by Argyres and Seiberg to be S-dual, namely the SU(3) gauge theory with six quarks, and the SU(2) gauge theory with one pair of quarks coupled to the superconformal theory with E_6 flavor symmetry. In mathematical terms, we demonstrate the equivalence between a hyperkähler quotient of a linear space and another hyperkähler quotient involving the minimal nilpotent orbit of E_6, modulo the identification of the twistor lines.

Keywords: S-duality, nilpotent orbit, hyperkähler quotient.

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1. Introduction

1.1 Argyres-Seiberg duality

In a remarkable paper [1], a new type of strong-weak duality of four-dimensional $\mathcal{N}=2$ theories was introduced. Consider an $\mathcal{N}=2$ supersymmetric SU(3) gauge theory with six quarks in the fundamental representation. This theory has vanishing one-loop beta function, and the gauge coupling constant

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2} \tag{1.1}$$

is exactly marginal. Argyres and Seiberg carried out a detailed study of the behavior of the Seiberg-Witten curve close to the point $\tau \to 1$ where the theory is infinitely strongly-coupled, and were led to conjecture a dual description involving an SU(2) group with gauge coupling

$$\tau' = \frac{1}{1-\tau}. \tag{1.2}$$

To understand the matter content of the dual theory, one first needs to recall the interacting superconformal field theory (SCFT) with flavor symmetry E_6 first described by [2]. This theory has one-dimensional Coulomb branch parametrized by u whose scaling dimension is 3, and is realized as the low-energy limit of the worldvolume theory on a D3-brane probing the transverse geometry of an F-theory 7-brane with E_6 gauge group. The gauge group on the 7-brane then manifests as a flavor symmetry from the point of view of the D3-brane. We denote this theory by SCFT$[E_6]$ following [1].

Now, the theory Argyres and Seiberg proposed as the dual of the SU(3) gauge theory with six quarks consists of the SU(2) gauge bosons, coupled to one hypermultiplet in the doublet representation, and also to a subgroup $\mathrm{SU}(2) \subset E_6$ of SCFT$[E_6]$. The SU(2) subgroup is chosen so that the raising operator of SU(2) maps to the raising operator for the highest root of E_6. In the following, we refer to two sides of the duality as the SU(3) side and the exceptional side, respectively.

Argyres and Seiberg provided a few compelling pieces of evidence for this duality. First, the flavor symmetry agrees. On the SU(3) side, there is a $\mathrm{U}(6) = \mathrm{U}(1) \times \mathrm{SU}(6)$ symmetry which rotates the six quarks. On the exceptional side, there is an SO(2) symmetry which rotates a pair of quarks in the doublet representation, which can be identified with the U(1) part of U(6). Then, the flavor symmetry of the SCFT with E_6 is broken down to the maximal subgroup commuting with $\mathrm{SU}(2) \subset E_6$, which is SU(6). Second, the scaling dimensions of Coulomb-branch operators agree. Indeed, on the SU(3) side one has $\operatorname{tr}\phi^2$ and $\operatorname{tr}\phi^3$ where ϕ is the adjoint chiral multiplet of SU(3). The dimensions are thus 2 and 3. On the exceptional side, one has $\operatorname{tr}\varphi^2$ (where φ is

the adjoint chiral multiplet of SU(2)), which has dimension 2, and the Coulomb-branch operator u of SCFT$[E_6]$, which has dimension 3.

Third, Argyres and Seiberg studied in detail the deformation of the Seiberg-Witten curve under the SU(6) mass deformation, and found remarkable agreement. Fourth, they computed the current algebra central charge of the SU(6) flavor symmetry on the SU(3) side, which agreed with the central charge of the E_6 symmetry on the exceptional side, inferred from the fact that the beta function of the SU(2) gauge group coupling is zero. This is as it should be, because SU(6) arises as a subgroup of E_6 on the exceptional side. This provided a prediction of the current central charge of SCFT$[E_6]$ for the first time, which was later reproduced holographically by [3]. There are generalizations to similar duality pairs involving SCFTs with flavor symmetries other than E_6 [1, 4].

Our aim in this note is to present further convincing evidence for this duality, by showing that the Higgs branches of the two sides of the duality are equivalent as hyperkähler cones. Mathematically speaking, we will show the agreement of their twistor spaces as complex varieties with real structure, but we have not been able to prove that they share the same family of twistor lines. Instead we give numerical evidence that their Kähler potentials agree in Appendix C.

1.2 Higgs branch

On the SU(3) side, let us denote the squark fields by

$$Q^i_a, \qquad \widetilde{Q}^a_i \tag{1.3}$$

where $i = 1, \ldots, 6$ are the flavor indices and $a = 1, 2, 3$ the color indices. The Higgs branch is the locus where the F-term and the D-term both vanish, divided by the action of the gauge group SU(3). As is well known, this space can also be obtained by setting $F = 0$ without setting $D = 0$, and dividing by the complexified gauge group $\mathrm{SL}(3, \mathbb{C})$.

Thus the Higgs branch is parametrized by gauge invariant composite operators

$$M^i{}_j = Q^i_a \widetilde{Q}^a_j, \qquad B^{ijk} = \epsilon^{abc} Q^i_a Q^j_b Q^k_c, \qquad \widetilde{B}_{ijk} = \epsilon_{abc} \widetilde{Q}^a_i \widetilde{Q}^b_j \widetilde{Q}^c_k \tag{1.4}$$

which satisfy various constraints, e.g.

$$B^{[ijk} M^{l]}{}_m = 0 \tag{1.5}$$

to which we will come back later. The fields Q^i_a, $\widetilde{Q}^a_i$ have 36 complex components, while the F-term condition imposes 8 complex constraints. The quotient by $\mathrm{SL}(3, \mathbb{C})$ reduces the complex dimension further by 8, so the Higgs branch has complex dimension

$$2 \times 3 \times 6 - 8 - 8 = 20. \tag{1.6}$$

Our problem is to understand how this structure of the Higgs branch is realized on the exceptional side. Firstly, we have one hypermultiplet in the doublet representation, which we denote as v_α, $\widetilde{v}_\alpha$ in $\mathcal{N}=1$ superfield notation. Here $\alpha=1,2$ is the doublet index.

We also have the Higgs branch of SCFT$[E_6]$, the structure of which is known through the F-theoretic construction of the SCFT. Recall that this theory is the worldvolume theory on one D3-brane probing a F-theory 7-brane of type E_6. Say the D3-brane extends along the directions 0123, and the 7-brane along the directions 01234567. The one-dimensional Coulomb branch of this theory is identified with the transverse directions 89 to the 7-brane. The theory becomes superconformal when the D3-brane hits the 7-brane, at which point the Higgs branch emanates. This is identified as the process where a D3-brane is absorbed into the worldvolume of the 7-brane as an E_6 instanton along the directions 4567. The real dimension of the N-instanton moduli space of E_6 is $4h_{E_6}N$ with the dual Coxeter number $h_{E_6}=12$. The center-of-mass motion of the instanton corresponds to a decoupled free hypermultiplet, and thus the genuine moduli space is the so-called 'centered' one-instanton moduli space without the center-of-mass motion, which has complex dimension 22.

The SU(2) gauge group couples to the quark fields v_α, $\widetilde{v}_\alpha$, and this instanton moduli space. Imposing the F-term condition and dividing by the complexified gauge group, we find the complexified dimension of the Higgs branch as

$$2\times 2+22-3-3=20, \tag{1.7}$$

which correctly reproduces the dimension of the Higgs branch on the SU(3) side.

We would like to perform more detailed checks, and for that purpose one needs to have a concrete description of the instanton moduli. It is well known that the ADHM description is available for classical gauge groups, but how can we proceed for exceptional groups? Luckily, there is another description of the 1-instanton moduli spaces, applicable to any group G, which identifies the centered 1-instanton moduli space with the minimal nilpotent orbit of G [5].

Let us now define the minimal nilpotent orbit. $G_\mathbb{C}$ acts on the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$, which has the Cartan generators H^i and the raising/lowering operators $E_{\pm\rho}$ for roots ρ. $G_\mathbb{C}$ also acts on the dual vector space $\mathfrak{g}_\mathbb{C}^*$ of $\mathfrak{g}_\mathbb{C}$ via the coadjoint action,[1] and the minimal nilpotent orbit $\mathcal{O}_{\min}(G)$ of G is the orbit of $(E_\theta)^*$, where θ denotes the highest root:

$$\mathcal{O}_{\min}(G)=G_\mathbb{C}\cdot(E_\theta)^*. \tag{1.8}$$

[1] One can of course identify $\mathfrak{g}_\mathbb{C}^*$ and $\mathfrak{g}_\mathbb{C}$ using the Killing form, but it is more mathematically natural to use the coadjoint representation here.

The minimal nilpotent orbit is known to have polynomial defining equations. Moreover, they can be chosen to be quadratic, transforming covariantly under $G_{\mathbb{C}}$. These relations are known under the name of the Joseph ideal [6]. The simplest example is the case $G = \mathrm{SU}(2)$. In this case $\mathfrak{g}_{\mathbb{C}}$ is three-dimensional; denote its three coordinates by a, b and c, which transform as a triplet of SU(2). The minimal nilpotent orbit is then given by

$$a^2 + b^2 + c^2 = 0 \tag{1.9}$$

which describes the space $\mathbb{C}^2/\mathbb{Z}_2$, and as is well-known, the centered one-instanton moduli space of SU(2) is exactly this orbifold.

Let us come back to the case of E_6. We fix an SU(2) subalgebra generated by $E_{\pm\theta}$. The maximal commuting subalgebra is then SU(6). The E_6 algebra can be decomposed under the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(6)$ into

$$X^i{}_j, \qquad Y_\alpha^{[ijk]}, \qquad Z_{\alpha\beta} \tag{1.10}$$

where $i, j, k = 1, \ldots, 6$ are the SU(6) indices, $\alpha, \beta = 1, 2$ those for SU(2). Here $X^i{}_j$ and $Z_{\alpha\beta}$ are adjoints of SU(6) and SU(2) respectively, and Y_α^{ijk} transforms as the three-index anti-symmetric tensor of SU(6) times the doublet of SU(2). The minimal nilpotent orbit is then given by the simultaneous zero locus of quadratic equations in X, Y and Z which we describe in detail later.

For now let us see what are the gauge-invariant coordinates of the Higgs branch of the exceptional side. The SU(2) gauge group is identified to the $\mathrm{SU}(2) \subset E_6$ just chosen above, i.e. the SU(2) gauge bosons couple to the current of this SU(2) subgroup of the E_6 symmetry. We also have the quarks v_α and $\widetilde{v}_\alpha$ in addition to the fields X, Y and Z, and we need to make SU(2)-invariant combinations of them. Moreover, we need to impose the F-term equation, which is

$$Z_{\alpha\beta} + v_{(\alpha}\widetilde{v}_{\beta)} = 0 \tag{1.11}$$

as we argue later. Thus, any appearance of Z inside a composite operator can be eliminated in favor of v and $\widetilde{v}$. Therefore we have the following natural gauge-invariant composites, from which all gauge-invariant operators can be generated as will be shown in Sec. 5.4:

$$(v\widetilde{v}), \qquad X^i{}_j, \qquad (Y^{ijk}v), \qquad (Y_{ijk}\widetilde{v}). \tag{1.12}$$

Here we defined

$$(uw) \equiv u_\alpha w_\beta \epsilon^{\alpha\beta} \tag{1.13}$$

for two doublets u_α and w_α, and $Y_{ijk,\alpha}$ is defined by lowering the indices of Y_α^{ijk} by the epsilon tensor, see Appendix A.

This suggests the following identifications between the operators on the two sides of the duality:

$$\operatorname{tr} M \leftrightarrow (v\widetilde{v}), \qquad \hat{M}^i{}_j \leftrightarrow X^i{}_j, \tag{1.14}$$

$$B^{ijk} \leftrightarrow (Y^{ijk}v), \qquad \widetilde{B}_{ijk} \leftrightarrow (Y_{ijk}\widetilde{v}) \tag{1.15}$$

where $\hat{M}^i{}_j$ is the traceless part of $M^i{}_j$. The identifications preserve the dimensions of the operators if we assign dimensions 2 to the fields X, Y and Z. The SU(6) transformation nicely agrees. The U(1) part of the flavor symmetry can be matched if one assigns charge ± 1 to Q, $\widetilde{Q}$, and charge ± 3 to v, $\widetilde{v}$.

This factor of 3 was predicted in the original paper [1] from a totally different point of view, by demanding that the the two-point function of two U(1) currents should agree under the duality. Let us quickly recall how it was derived there. The form of the two-point function of the U(1) current j_μ is strongly constrained by the conservation and the conformal symmetry, and we have

$$\langle j_\mu(x) j_\nu(0)\rangle \propto k\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \cdots . \tag{1.16}$$

k is called the central charge, and $\cdots$ stands for less singular terms. Let us normalize k such that one hypermultiplet of charge q contributes q^2 to k. Assign Q, $\widetilde{Q}$ the charge ± 1, and let the charge of v, $\widetilde{v}$ be $\pm q$. Then k calculated from the SU(3) side is $6 \times 3 = 18$, while k determined from the exceptional side is $2q^2$. Equating these, Argyres and Seiberg concluded that the charge of v, $\widetilde{v}$ should be $q = \pm 3$.

The agreement is already impressive at this stage, but we would like to see how the constraints are mapped. We would also like to study how the hyperkähler structures agree, because so far we considered the Higgs branch only as a complex manifold. For that purpose we need to recall more about the hyperkähler cone.

The structure of the rest of the paper is as follows: We discuss in Sec. 2 what data are mathematically necessary to show the equivalence of the Higgs branches. Sec. 3 is devoted to the description of the minimal nilpotent orbit, i.e. the 1-instanton moduli space, as a hyperkähler space. Sec. 4 and Sec. 5 will be spent in calculating the necessary data on the SU(3) side and the exceptional side, respectively. Then they are compared in Sec. 6 which shows remarkable agreement. We conclude in Sec. 7. We have four Appendices: Appendix A collects our conventions, Appendix B gathers the machinery of twistor spaces required to show the equivalence of hyperkähler cones, and Appendix C compares the Kähler potentials of the duality pair. Appendix D is a summary for mathematicians.

2. Rudiments of hyperkähler cones

Here we collect the basics of the hyperkähler cones in a physics language. Mathematically precise formulation can be found in [7, 8]. The Higgs branch $\mathcal{M}$ of an $\mathcal{N}=2$ gauge theory is a hyperkähler manifold, i.e. one has three complex structures $J^{1,2,3}$ satisfying $J^1J^2=J^3$, compatible with the metric g, and the associated two-forms $\omega_{1,2,3}$ are all closed. We choose a particular $\mathcal{N}=1$ supersymmetry subgroup of the $\mathcal{N}=2$ supersymmetry group, which distinguishes one of the complex structures, say $J\equiv J^3$. $\mathcal{M}$ is then thought of as a Kähler manifold with the Kähler form $\omega=\omega_3$. $\Omega=\omega_1+i\omega_2$ is a closed $(2,0)$-form on $\mathcal{M}$ which then defines a holomorphic symplectic structure on $\mathcal{M}$. Physically this means that the $\mathcal{N}=1$ chiral ring, i.e. the ring of holomorphic functions on $\mathcal{M}$, has a natural holomorphic Poisson bracket

$$[f_1,f_2]=(\Omega^{-1})^{ij}\partial_i f_1\partial_j f_2 \tag{2.1}$$

for two holomorphic functions $f_{1,2}$ on $\mathcal{M}$.

Second, we are dealing with the Higgs branch of an $\mathcal{N}=2$ superconformal theory, which has the dilation and the $\mathrm{SU}(2)_R$ symmetry built in the symmetry algebra. The dilation makes $\mathcal{M}$ into a cone with the metric

$$ds^2_{\mathcal{M}}=dr^2+r^2ds^2_{\text{base}}, \tag{2.2}$$

and $\mathrm{SU}(2)_R$ symmetry acts on the base of the cone as an isometry, rotating the three complex structures as a triplet. These two conditions make $\mathcal{M}$ into a hyperkähler cone. $K=r^2$ is a Kähler potential with respect to any of the complex structures $J^{1,2,3}$, and is called the hyperkähler potential in the mathematical literature. The dilatation assigns the scaling dimensions, or equivalently the weights, to the chiral operators on $\mathcal{M}$.

Let us consider an element of $\mathrm{SU}(2)_R$ which acts on the three complex structures as $(J^1,J^2,J^3)\to(J^1,-J^2,-J^3)$. This element defines an anti-holomorphic involution $\sigma:\mathcal{M}\to\mathcal{M}$ because it reverses $J\equiv J^3$. This induces an operation σ^* on holomorphic functions on $\mathcal{M}$ via $(\sigma^*(f))(x)\equiv f(\sigma(x))$. σ^* maps holomorphic functions to anti-holomorphic functions, but is a linear operation, not a conjugate-linear operation. We call this operation the conjugation.

As will be detailed in Appendix B, the space $\mathcal{M}$ as a complex manifold, with the Poisson brackets, the scaling weights and the conjugation, almost suffices to reconstruct the hyperkähler metric on $\mathcal{M}$. Therefore, our main task in checking the agreement of the Higgs branches of the duality pair is to identify them as complex manifolds, and to show that the extra data defined on them also coincide. In order to complete the proof we need to show that the families of the twistor lines coincide, which we have not been

able to do. Instead we will give numerical support by calculating the Kähler potential directly on both sides in Appendix C.

The Higgs branches $\mathcal{M}$ that we treat here are gauge theory moduli spaces. They can be described by the hyperkähler quotient construction [9], which we now review. Let us start with an $\mathcal{N}=2$ gauge theory with the gauge group G, whose hypermultiplets take value in the hyperkähler manifold $\mathcal{X}$. The action of G on $\mathcal{X}$ preserves three Kähler structures, and thus there are three moment maps μ_s^a ($s=1,2,3$; $a=1,\ldots,\dim G$) which satisfy

$$d\mu_s^a = \iota_{\xi^a}\omega_s \tag{2.3}$$

where ξ^a is the Killing vector associated to the a-th generator of G. The Higgs branch of the gauge theory, in the absence of any non-zero Fayet-Iliopoulos parameter, is then given by

$$\mathcal{M} \equiv \mathcal{X}/\!\!/\!\!/G \equiv \{x\in\mathcal{X} \mid \mu_s^a(x)=0\}/G. \tag{2.4}$$

With one complex structure $J=J^3$ chosen, it is convenient to call

$$D^a=\mu_3^a, \qquad F^a=\mu_1^a+i\mu_2^a. \tag{2.5}$$

Then, as a complex manifold,

$$\mathcal{M}=\{x\in\mathcal{X} \mid F^a=0\}/G_{\mathbb{C}}. \tag{2.6}$$

It is instructive to note that F^a is exactly the Hamiltonian which generates the G action on the chiral ring of $\mathcal{X}$, under the Poisson bracket associated to $\Omega=\omega_1+i\omega_2$.

The conjugation σ^* and the Poisson bracket $[\cdot,\cdot]$ on the quotient $\mathcal{M}$ are given by the restriction of the corresponding operations on $\mathcal{X}$. It is instructive to see why the Poisson bracket of the quotient is well-defined: two G-invariant holomorphic functions $f_{1,2}$ on $\mathcal{X}$ lead to the same function on $\mathcal{M}$ if and only if $f_1=f_2+u_aF^a$ with holomorphic functions u_a. Then we have, for a G-invariant holomorphic function h,

$$[f_1,h]-[f_2,h]=[u_aF^a,h]=[u_a,h]F^a+u_a[F^a,h] \tag{2.7}$$

on $\mathcal{X}$. The first term in the right hand side is zero on $\mathcal{M}$ because we set $F^a=0$, while the second term is zero because h is G-invariant. Therefore $[f_1,h]$ and $[f_2,h]$ determine the same holomorphic function on $\mathcal{M}$.

The Kähler potential of $\mathcal{M}$ is similarly the restriction of that of $\mathcal{X}$ to the zero locus of the moment maps in our situation, as discussed in Sec. 2B of [9]. To illustrate the procedure, let us consider an $\mathcal{N}=1$ supersymmetric U(1) gauge theory coupled to chiral fields Φ_i of charge q_i whose Lagrangian is

$$L=\int d^4\theta\,(\sum_i \Phi_i^* e^{2q_iV}\Phi_i+\xi V) \tag{2.8}$$

where ξ is the Fayet-Iliopoulos parameter. The moduli space can be determined by taking the gauge coupling to be formally infinite, i.e. treating the linear superfield V as an auxiliary field. Then V is determined via its equation of motion

$$\sum_i q_i \Phi_i^* e^{2q_i V} \Phi_i + \xi = 0, \tag{2.9}$$

i.e. $\Phi_i' = e^{q_i V}\Phi_i$ solve the usual D-term equation. The Kähler potential of the moduli space is then given by plugging the solution to (2.9) into (2.8). It can be generalized to any gauge group, and the result agrees with the mathematical formula given in Sec. 3.1 of [10]. This shows that the Kähler potential is given just by the restriction of the original one if $\xi = 0$. This analysis does not incorporate quantum corrections, but it is well-known that for $\mathcal{N} = 2$ theories the quantum effect does not modify the hyperkähler structure, see Sec. 3 of [11].

3. Geometry of the minimal nilpotent orbit

Here we gather the relevant information on the hyperkähler geometry of the minimal nilpotent orbit of any simple group G, which coincides with the centered moduli space of single instantons with gauge group G [5, 8]. We hope this section might be useful for anyone who wants to deal with the one-instanton moduli space. In the following G stands for a compact simple Lie group, $\mathfrak{g}_\mathbb{R}$ its Lie algebra. We let $G_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$ be complexifications of G and $\mathfrak{g}_\mathbb{R}$ respectively.

The existence of a uniform description of the one-instanton moduli space applicable to any G might be understood as follows: we can construct a one-instanton configuration easily by taking a BPST instanton of SU(2) and regard it as an instanton of G via a group embedding $\mathrm{SU}(2) \subset G$. It is known that any one-instanton of G arises in this manner [12, 13]. The one-instanton moduli space is then parameterized by the position, the size, and the gauge orientation of the BPST instanton inside G. This description realizes the one-instanton moduli space as a cone over a homogeneous manifold G/H, where H is the maximal subgroup of G which commutes with the SU(2) used in the embedding. It is however not directly suitable for the analysis of its complex structure. For that purpose we use another realization of the one-instanton moduli space as the minimal nilpotent orbit $\mathcal{O}_{\min}$ of G [5].

Let us define $\mathcal{O}_{\min}$. First we decompose $\mathfrak{g}_\mathbb{C}$ into the Cartan generators H^i and the raising/lowering operators $E_{\pm\rho}$ for roots ρ. The minimal nilpotent orbit $\mathcal{O}_{\min}(G)$ is then the orbit of $(E_\theta)^*$ in $\mathfrak{g}_\mathbb{C}^*$, where θ denotes the highest root:

$$\mathcal{O}_{\min}(G) = G_\mathbb{C} \cdot (E_\theta)^* \subset \mathfrak{g}_\mathbb{C}^*. \tag{3.1}$$

We will write $\mathcal{O}_{\rm min}$ without explicitly writing G for the sake of simplicity when there is no confusion.

We think of elements of $\mathfrak{g}_\mathbb{C}$ as holomorphic functions on $\mathcal{O}_{\rm min}$, i.e. we have holomorphic functions[2] $\mathbb{X}^a$ ($a = 1, \ldots, \dim G$) on $\mathcal{O}_{\rm min}$. The defining equations of $\mathcal{O}_{\rm min}$ are a set of quadratic equations which we call the Joseph relations [6].[3]

These relations can be studied using a theorem of Kostant [14]: Let $V(\alpha)$ denote the representation space of a semisimple group G with the highest weight α, and let $v \in V(\alpha)^*$ be a vector in the highest weight space. The orbit $G_\mathbb{C} \cdot v$ is then an affine algebraic variety whose defining ideal $\mathcal{I}$ is generated by its degree-two part $\mathcal{I}_2$. Furthermore, $\mathcal{I}_2$ is given by the relation

$$\mathrm{Sym}^2 V(\alpha) = V(2\alpha) \oplus \mathcal{I}_2 \tag{3.2}$$

where we identify $\mathrm{Sym}^2 V(\alpha)$ as the space of degree-two polynomials on $V(\alpha)^*$. The minimal nilpotent orbit is exactly of this form where $V(\alpha)$ is the adjoint representation, i.e.

$$\mathcal{O}_{\rm min} = \{\mathbb{X} \in \mathfrak{g}^*_\mathbb{C} \bigm| (\mathbb{X} \otimes \mathbb{X})|_{\mathcal{I}_2} = 0\}. \tag{3.3}$$

For practice, let us apply this to the case $G = \mathrm{SU}(2)$. There, $V(\alpha)$ is the triplet representation, so by (3.2) $\mathcal{I}_2$ is the singlet representation. Therefore, if we parameterize $\mathfrak{su}(2)$ by (a, b, c), the minimal nilpotent orbit is given by the equation

$$a^2 + b^2 + c^2 = 0, \tag{3.4}$$

which is $\mathbb{C}^2/\mathbb{Z}_2$ as it should be.

Now that we have given $\mathcal{O}_{\rm min}$ as a complex manifold, let us describe its hyperkähler structure. The main fact we use is that G acts isometrically on $\mathcal{O}_{\rm min}$, preserving the hyperkähler structure.

There is a triplet of moment maps μ^a_s for this action where $a = 1, \ldots, \dim G$ and $s = 1, 2, 3$. The functions $\mathbb{X}^a$ are the holomorphic moment maps of the G action, i.e. $\mathbb{X}^a = \mu^a_1 + i\mu^a_2$, It follows that their Poisson bracket is

$$[\mathbb{X}^a, \mathbb{X}^b] = f^{ab}{}_c \mathbb{X}^c \tag{3.5}$$

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[2]More mathematically, one has a natural holomorphic $\mathfrak{g}^*_\mathbb{C}$-valued function $\mathbb{X} : \mathcal{O}_{\rm min} \hookrightarrow \mathfrak{g}^*_\mathbb{C}$ given by the embedding. Then every element $t \in \mathfrak{g}_\mathbb{C}$ gives a holomorphic function $(\mathbb{X}, t)$ on $\mathcal{O}_{\rm min}$ via $x \in \mathcal{O}_{\rm min} \mapsto (\mathbb{X}(x), t)$. Our $\mathbb{X}^a$ is $(\mathbb{X}, T^a)$ for a generator T^a of $\mathfrak{g}_\mathbb{C}$. We take a real basis of $\mathfrak{g}_\mathbb{C}$, so in fact $T^a \in \mathfrak{g}_\mathbb{R} \subset \mathfrak{g}_\mathbb{C}$.

[3]Strictly speaking, the Joseph ideal is a two-sided ideal in the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, and what we use below is its associated ideal in the polynomial algebra.

where ${f^{ab}}_c$ are the structure constants of G. Phrased differently, the holomorphic symplectic structure underlying the hyperkähler structure of the nilpotent orbit is the standard Kirilov–Kostant–Souriau symplectic form on the coadjoint orbit [5, 8].

The conjugation is given by the $\mathrm{SU}(2)_R$ action, which sends (μ_1, μ_2, μ_3) to $(\mu_1, -\mu_2, -\mu_3)$. Therefore

$$\sigma^*(\mathbb{X}^a) = (\mathbb{X}^a)^*. \tag{3.6}$$

The scaling dimension of $\mathbb{X}$ is fixed to be two, as it should be for the F-term in an $\mathcal{N} = 2$ supersymmetric theory.

Let us next describe a Kähler potential for $\mathcal{O}_{\min}$, which was determined in [15]. The derivation boils down to the following: G acts on $\mathcal{O}_{\min}$ with cohomogeneity one, and by averaging over this action we can consider K to be G-invariant; so K is a function of $\operatorname{tr}\mathbb{X}\mathbb{X}^*$. K should be of scaling dimension two, so that K is proportional to $\sqrt{\operatorname{tr}\mathbb{X}\mathbb{X}^*}$ up to a constant. The constant factor can be fixed by considering a particular element on $\mathcal{O}_{\min}$. For this purpose we again turn to the minimal nilpotent orbit of $\mathrm{SU}(2)$, which is $\mathbb{C}^2/\mathbb{Z}_2$. The normalization of the Kähler potential of the minimal nilpotent orbit of a general group can then be determined because it contains the minimal nilpotent orbit of $\mathrm{SU}(2)$ as a subspace.

We parameterize $\mathbb{C}^2$ by $(u, \widetilde{u})$ and divide by the multiplication by -1. We define our conventions for the holomorphic Poisson bracket and the Kähler potential of a flat $\mathbb{H}$ as follows:

$$K = |u|^2 + |\widetilde{u}|^2, \qquad [u, \widetilde{u}] = 1. \tag{3.7}$$

Now, $\mathbb{C}^2/\mathbb{Z}_2$ is parametrized by

$$Z_{11} = u^2/2, \quad Z_{12} = Z_{21} = u\widetilde{u}/2, \quad Z_{22} = \widetilde{u}^2/2 \tag{3.8}$$

which satisfy

$$Z_{11}Z_{22} = {Z_{12}}^2. \tag{3.9}$$

The Kähler potential is now

$$K = 2\sqrt{|Z_{11}|^2 + |Z_{22}|^2 + 2|Z_{12}|^2} = 2\sqrt{Z_{\alpha\beta}\overline{Z}^{\alpha\beta}}. \tag{3.10}$$

Then, the moment map associated to the generator J_3 of non-R $\mathrm{SU}(2)$ acting on $\mathbb{C}^2/\mathbb{Z}_2$ can be explicitly calculated, with the result

$$F = Z_{12}, \quad D = \frac{2}{K}(|Z_{11}|^2 - |Z_{22}|^2). \tag{3.11}$$

Now that the preparation is done, we move on to the calculation of the Higgs branch on both sides of the duality.

4. SU(3) side

The theory has six quarks in the fundamental representation,

$$Q^i_a, \qquad \widetilde{Q}^a_i \tag{4.1}$$

where $a = 1, \ldots, 6$ and $i = 1, 2, 3$. As is well known, any SU(3)-invariant polynomial constructed out of these fields is a polynomial in the operators [11]:

$$M^i{}_j = Q^i_a \widetilde{Q}^a_j, \qquad B^{ijk} = \epsilon^{abc} Q^i_a Q^j_b Q^k_c, \qquad \widetilde{B}_{ijk} = \epsilon_{abc} \widetilde{Q}^a_i \widetilde{Q}^b_j \widetilde{Q}^c_k. \tag{4.2}$$

In the following we study the Poisson brackets, the action of the conjugation, and the constraints in turn.

4.1 Poisson brackets

The Poisson bracket of the basic fields is given by

$$[Q^i_a, \widetilde{Q}^b_j] = \delta^i{}_j \delta^b{}_a. \tag{4.3}$$

Then we have, for example,

$$[M^i{}_j, Q^k_a] = -\delta^k{}_j Q^i_a, \tag{4.4}$$

i.e. $M^i{}_j$ is the generator of U(6). We define $\operatorname{tr} M$ to be the trace of $M^i{}_j$, and

$$\hat{M}^i{}_j = M^i{}_j - \frac{1}{6} \delta^i{}_j \operatorname{tr} M \tag{4.5}$$

is its traceless part. $\hat{M}^i{}_j$ is the SU(6) generator and $\operatorname{tr} M$ the U(1) generator. We define the U(1) charge q of an operator $\mathcal{O}$ to be given by

$$[\operatorname{tr} M, \mathcal{O}] = -q\mathcal{O}. \tag{4.6}$$

The most complicated bracket is

$$\begin{aligned}[B^{ijk}, \widetilde{B}_{lmn}] &= 18 M^{[i}{}_{[l} M^j{}_m \delta^{k]}{}_{n]} \\ &= 18 \hat{M}^{[i}{}_{[l} \hat{M}^j{}_m \delta^{k]}{}_{n]} + 6(\operatorname{tr} M) \hat{M}^{[i}{}_{[l} \delta^j{}_m \delta^{k]}{}_{n]} + \frac{1}{2} (\operatorname{tr} M)^2 \delta^{[i}{}_{[l} \delta^j{}_m \delta^{k]}{}_{n]}. \end{aligned} \tag{4.7}$$

4.2 Conjugation

We choose the involution on the elementary fields to be

$$\sigma^*(Q^i_a) = (\widetilde{Q}^a_i)^*, \qquad \sigma^*(\widetilde{Q}^a_i) = -(Q^i_a)^*. \tag{4.8}$$

Then the transformation of the composites are

$$\sigma^*(M^i{}_j) = -(M^j{}_i)^*, \qquad \sigma^*(\operatorname{tr} M) = -(\operatorname{tr} M)^*, \tag{4.9}$$

$$\sigma^*(B^{ijk}) = (\widetilde{B}_{ijk})^*, \qquad \sigma^*(\widetilde{B}_{ijk}) = -(B^{ijk})^*. \tag{4.10}$$

4.3 Constraints

The constraints were studied in [11]. Those which come before imposing the F-term constraint are

$$B^{ijk}\widetilde{B}_{lmn} = 6M^{[i}{}_{l}M^{j}{}_{m}M^{k]}{}_{n}, \tag{4.11}$$

$$B^{ij[k}B^{lmn]} = 0, \qquad \widetilde{B}_{ij[k}\widetilde{B}_{lmn]} = 0, \tag{4.12}$$

$$M^{[i}{}_{j}B^{klm]} = 0, \qquad M^{i}{}_{[j}\widetilde{B}_{klm]} = 0. \tag{4.13}$$

The F-term constraint

$$Q^i_a\widetilde{Q}^b_i - \frac{1}{3}\delta^b_a(Q\widetilde{Q}) = 0 \tag{4.14}$$

further imposes

$$\hat{M}^{i}{}_{j}B^{jkl} = \frac{1}{6}(\mathrm{tr}\, M)B^{ikl}, \tag{4.15}$$

$$\hat{M}^{i}{}_{j}\widetilde{B}_{ikl} = \frac{1}{6}(\mathrm{tr}\, M)\widetilde{B}_{jkl}, \tag{4.16}$$

$$\hat{M}^{i}{}_{j}M^{j}{}_{k} = \frac{1}{6}(\mathrm{tr}\, M)M^{i}{}_{k}. \tag{4.17}$$

We will find it convenient later to have constraints in terms of irreducible representations (irreps) of SU(6). We use the Dynkin labels to distinguish the irreps in the following. The $MB = 0$ relations (4.13), (4.15), (4.16) give

$$\hat{M}^{\{i}{}_{l}B^{[jk]\}l} = 0, \qquad \hat{M}^{\{i}{}_{l}\widetilde{B}^{[jk]\}l} = 0, \tag{4.18}$$

$$\hat{M}^{l}{}_{\{i}B_{[jk]\}l} = 0, \qquad \hat{M}^{l}{}_{\{i}\widetilde{B}_{[jk]\}l} = 0. \tag{4.19}$$

Here we defined the projector from a tensor with the structure $A_{i[jk]}$ to the irrep $(1,1,0,0,0)$ by $A_{\{i[jk]\}} \equiv A_{i[jk]} - A_{[i[jk]]}$. We also have

$$\hat{M}^{[i}{}_{l}B^{jk]l} = \frac{1}{6}(\mathrm{tr}\, M)B^{ijk}, \qquad \hat{M}^{l}{}_{[i}\widetilde{B}_{jk]l} = \frac{1}{6}(\mathrm{tr}\, M)\widetilde{B}_{ijk}. \tag{4.20}$$

The $MM = 0$ relation (4.17) gives

$$\hat{M}^{i}{}_{j}\hat{M}^{j}{}_{k} = \frac{1}{6}\delta^{i}{}_{k}\hat{M}^{m}{}_{n}\hat{M}^{n}{}_{m}, \tag{4.21}$$

$$\hat{M}^{i}{}_{j}\hat{M}^{j}{}_{i} = \frac{1}{6}(\mathrm{tr}\, M)^2. \tag{4.22}$$

The $BB = 0$ relation (4.12) gives

$$B^{ikl}B_{jkl} = 0, \qquad \widetilde{B}^{ikl}\widetilde{B}_{jkl} = 0. \tag{4.23}$$

Finally, the decomposition of the $B\widetilde{B} = MMM$ relation gives, using (4.21) and (4.22) repeatedly,

$$B^{ijk}\widetilde{B}_{ijk} = \frac{2}{9}(\mathrm{tr}\, M)^3, \tag{4.24}$$

$$B^{ikl}\widetilde{B}_{jkl}\,\big|_{\mathbf{adj}} = \frac{2}{9}(\mathrm{tr}\, M)^2 \hat{M}^i{}_j, \tag{4.25}$$

$$B^{ijm}\widetilde{B}_{klm}\,\big|_{0,1,0,1,0} = \frac{2}{3}(\mathrm{tr}\, M)\hat{M}^{[i}{}_{[k}\hat{M}^{j]}{}_{l]}\,\big|_{0,1,0,1,0}, \tag{4.26}$$

$$B^{ijk}\widetilde{B}_{lmn}\,\big|_{0,0,2,0,0} = 6\hat{M}^{[i}{}_l\hat{M}^j{}_m\hat{M}^{k]}{}_n\,\big|_{0,0,2,0,0}\,. \tag{4.27}$$

5. Exceptional side

5.1 Poisson brackets

We have chiral fields $\mathbb{X}^a$ which transform in the adjoint of E_6, and satisfy the quadratic Joseph identities. We decompose $\mathbb{X}^a$ under the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(6) \subset E_6$. It gives

$$X^i{}_j, \quad Y^{[ijk]}_\alpha, \quad Z_{\alpha\beta} \tag{5.1}$$

where $X^i{}_j$ and $Z_{\alpha\beta}$ are the adjoints of SU(6) and SU(2) respectively, and Y^{ijk}_α is in the doublet of SU(2) and in the representation $(0, 0, 1, 0, 0)$, i.e. the three-index antisymmetric tensor, of SU(6). The Poisson brackets of the fields X, Y and Z are exactly the Lie brackets as explained above, which we take to be

$$[X^i{}_j, X^k{}_l] = \delta^i{}_l X^k{}_j - \delta^k{}_j X^i{}_l, \tag{5.2}$$

$$[Z_{\alpha\beta}, Z_{\gamma\delta}] = \frac{1}{2}(\epsilon_{\alpha\gamma}Z_{\beta\delta} + \epsilon_{\beta\gamma}Z_{\alpha\delta} + \epsilon_{\alpha\delta}Z_{\beta\gamma} + \epsilon_{\beta\delta}Z_{\alpha\gamma}) \tag{5.3}$$

and

$$[X^i{}_j, Y^{klm}_\alpha] = -3\delta^{[k}{}_j Y^{lm]i}_\alpha + \frac{1}{2}\delta^i{}_j Y^{klm}_\alpha, \tag{5.4}$$

$$[Z_{\alpha\beta}, Y^{ijk}_\gamma] = Y^{ijk}_{(\alpha}\epsilon_{\beta)\gamma} \tag{5.5}$$

and finally

$$[Y^{ijk}_\alpha, Y^{lmn}_\beta] = \epsilon^{ijklmn}Z_{\alpha\beta} - \frac{3}{2}\,\epsilon_{\alpha\beta}(X^{[i}{}_p\epsilon^{jk]lmnp} + X^{[l}{}_p\epsilon^{mn]ijkp}). \tag{5.6}$$

The final commutation relation can also be written as

$$[Y^{ijk}_\alpha, Y_{lmn\,\beta}] = -18X^{[i}{}_{[l}\delta^j{}_m\delta^{k]}{}_{n]} - 6Z_{\alpha\beta}\delta^{[i}{}_{[l}\delta^j{}_m\delta^{k]}{}_{n]}. \tag{5.7}$$

As we explained above, X, Y and Z are the holomorphic moment maps of the E_6 action. Therefore the contribution from $\mathcal{O}_{\rm min}$ to the F-term constraint for the SU(2) gauge group is given just by $Z_{\alpha\beta}$.

We take the bracket of v and $\widetilde{v}$ to be

$$[v_\alpha, \widetilde{v}_\beta] = \epsilon_{\alpha\beta}. \tag{5.8}$$

Then we have

$$[v_{(\alpha}\widetilde{v}_{\beta)}, v_\gamma] = v_{(\alpha}\epsilon_{\beta)\gamma}, \tag{5.9}$$

and

$$[(v\widetilde{v}), v_\alpha] = v_\alpha, \qquad [(v\widetilde{v}), \widetilde{v}_\alpha] = -\widetilde{v}_\alpha. \tag{5.10}$$

Recall that we define $(uw) \equiv u_\alpha w_\beta \epsilon^{\alpha\beta}$ for two doublets u_α and w_α. It is straightforward to check that $v_{(\alpha}\widetilde{v}_{\beta)}$ is the moment map of the SU(2) action on v and $\widetilde{v}$. Thus the F-term condition is

$$v_{(\alpha}\widetilde{v}_{\beta)} + Z_{\alpha\beta} = 0. \tag{5.11}$$

5.2 Conjugation

We take the conjugation on the variables $v, \widetilde{v}$ to be

$$\sigma^*(v_\alpha) = (\widetilde{v}_\beta)^* \epsilon_{\alpha\beta}, \qquad \sigma^*(\widetilde{v}_\alpha) = (v_\beta)^* \epsilon_{\alpha\beta}. \tag{5.12}$$

In terms of our variables $(X^i{}_j, Y_\alpha^{ijk}, Z_{\alpha\beta})$, the conjugation acts as follows:

$$\sigma^*(X^i{}_j) = -(X^j{}_i)^*, \tag{5.13}$$

$$\sigma^*(Y_\alpha^{ijk}) = (Y_{ijk\,\beta})^* \epsilon_{\alpha\beta}, \qquad \sigma^*(Y_{ijk\,\alpha}) = -(Y_\beta^{ijk})^* \epsilon_{\alpha\beta}, \tag{5.14}$$

$$\sigma^*(Z_{\alpha\beta}) = (Z_{\gamma\delta})^* \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}. \tag{5.15}$$

5.3 Constraints

As explained in Sec. 3, the Joseph relations are given by

$$(\mathbb{X} \otimes \mathbb{X})|_{\mathcal{I}_2} = 0 \tag{5.16}$$

where $\mathcal{I}_2$ is given by the relation

$$\mathrm{Sym}^2 V(\mathbf{adj}) = V(2\mathbf{adj}) \oplus \mathcal{I}_2. \tag{5.17}$$

Here, $V(\mathbf{adj})$ is the adjoint representation of E_6 whose Dynkin label is $\mathbf{adj} = \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$.

We then have

$$\mathcal{I}_2 = V\begin{pmatrix} & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \oplus V\begin{pmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.18}$$

The representations which appear in $\mathcal{I}_2$, decomposed under SU(2) × SU(6), are summarized in Table 1. The table reads as follows: e.g. for relation 4, the fourth column tells us there is one Joseph identity transforming as a doublet in SU(2) and as $(0,0,1,0,0)$ under SU(6), but the fifth column says one can construct two objects in this representation from bilinears in $X^i{}_j$, Y^{ijk}_α and $Z_{\alpha\beta}$. This means the identity has the form

$$0 = Y^{ijk}_\alpha Z_{\beta\gamma}\epsilon^{\alpha\beta} + c_4 X^{[i}{}_l Y^{jk]l}_\gamma, \tag{5.19}$$

where c_4 needs to be fixed, which can be done e.g. by explicitly evaluating the right hand side on a few elements on the nilpotent orbit. Elements on the nilpotent orbit can be readily generated, because one knows that the point

$$X^i{}_j = 0, \qquad Y^{ijk}_\alpha = 0, \qquad Z_{11} = 1, \qquad Z_{12} = Z_{22} = 0 \tag{5.20}$$

is on the nilpotent orbit by definition. Then the rest of the points can be generated by the coadjoint action of E_6, which can be obtained by exponentiating the structure constants.

Carrying out this program, we obtain the following full set of Joseph identities:

$$\begin{aligned}
&1. \qquad 0 = X^i{}_j Z_{\alpha\beta} + \frac{1}{4} Y^{ikl}_{(\alpha} Y_{jkl\beta)}, && (5.21)\\
&2. \qquad 0 = X^l{}_{\{i} Y_{[jk]\}l\alpha}, && (5.22)\\
&3. \qquad 0 = X^{\{i}{}_l Y^{[jk]\}l}_\alpha, && (5.23)\\
&4. \qquad 0 = Y^{ijk}_\alpha Z_{\beta\gamma}\epsilon^{\alpha\beta} + X^{[i}{}_l Y^{jk]l}_\gamma, && (5.24)\\
&5. \qquad 0 = (Y^{ijm}_\alpha Y_{klm\beta}\epsilon^{\alpha\beta} - 4X^{[i}{}_{[k} X^{j]}{}_{l]}) \big|_{0,1,0,1,0}, && (5.25)\\
&6. \qquad 0 = X^i{}_k X^k{}_j - \frac{1}{6}\delta^i{}_j X^k{}_l X^l{}_k, && (5.26)\\
&7. \qquad 0 = Y^{ijk}_\alpha Y_{ijk\beta}\epsilon^{\alpha\beta} + 24 Z_{\alpha\beta} Z_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}, && (5.27)\\
&7'. \qquad 0 = X^i{}_j X^j{}_i + 3 Z_{\alpha\beta} Z_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}. && (5.28)
\end{aligned}$$

	SU(2)	SU(6)	in $\mathcal{I}_2$	in $\mathrm{Sym}^2 V(\mathbf{adj})$
1.	**3**	$(1,0,0,0,1)$	1	2
2.	**2**	$(1,1,0,0,0)$	1	1
3.	**2**	$(0,0,0,1,1)$	1	1
4.	**2**	$(0,0,1,0,0)$	1	2
5.	**1**	$(0,1,0,1,0)$	1	2
6.	**1**	$(1,0,0,0,1)$	1	1
7.	**1**	$(0,0,0,0,0)$	2	3

Table 1: Decomposition of $\mathcal{I}_2$ in terms of SU(2) × SU(6) ⊂ E_6.

5.4 Gauge invariant operators

Let us enumerate the generators of the SU(2)-invariant operators constructed out of v_α, $\widetilde{v}_\alpha$, and $X^i{}_j$, Y^{ijk}_α, $Z_{\alpha\beta}$, using the F-term equation (5.11) and the Joseph identities (5.21) $\sim$ (5.28). Suppose we have a monomial constructed from those fields. We first replace every appearance of $Z_{\alpha\beta}$ by $-v_{(\alpha}\widetilde{v}_{\beta)}$. All the SU(2) indices are contracted by epsilon tensors of SU(2). Therefore the monomial is a product of $X^i{}_j$, $(v\widetilde{v})$, $(Y^{ijk}v)$, $(Y^{ijk}\widetilde{v})$ and $(Y^{ijk}Y^{lmn})$. The last of these can be eliminated using the Joseph identities. Indeed, the combination of the relations (5.25), (5.27) and (5.28) gives a Joseph identity of the form

$$Y^{ijk}_\alpha Y_{lmn\,\beta}\epsilon^{\alpha\beta} = 18X^{[i}{}_{[l}X^j{}_m\delta^{k]}{}_{n]} - 3Z_{\alpha\beta}Z_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\delta^{[i}{}_{[l}\delta^j{}_m\delta^{k]}{}_{n]}. \tag{5.29}$$

We conclude that any SU(2)-invariant polynomial is a polynomial in

$$X^i{}_j, \quad (v\widetilde{v}), \quad (Y^{ijk}v), \quad \text{and} \quad (Y^{ijk}\widetilde{v}). \tag{5.30}$$

6. Comparison

6.1 Identification of operators

Let us now proceed to the comparison of the structures we studied in Sec. 4 and in Sec. 5. We first make the following identification:

$$\hat{M}^i{}_j = X^i{}_j, \qquad \operatorname{tr} M = -3(v\widetilde{v}). \tag{6.1}$$

These are the moment maps of the flavor symmetries SU(6) and U(1), so the identification is fixed including the coefficients, and then the Poisson brackets involving either $\hat{M}$ or $\operatorname{tr} M$ automatically agree. The conjugation acting on $X^i{}_j$, $(v\widetilde{v})$ also agrees with that on $\hat{M}^i{}_j$ and $\operatorname{tr} M$.

We then set

$$B^{ijk} = c(Y^{ijk}v), \qquad \widetilde{B}_{ijk} = \widetilde{c}(Y_{ijk}\widetilde{v}). \tag{6.2}$$

One has

$$\sigma((Y^{ijk}v)) = (Y_{ijk}\widetilde{v})^*. \tag{6.3}$$

To be consistent with (4.10), we need to have

$$\widetilde{c} = c^*. \tag{6.4}$$

Let us then calculate the Poisson bracket of $(Y^{ijk}v)$ and $(Y_{lmn}\widetilde{v})$ using (5.29). We have

$$[(Y^{ijk}v), (Y_{lmn}\widetilde{v})] = -18X^{[i}{}_{[l}X^j{}_m\delta^{k]}{}_{n]} + 18(v\widetilde{v})X^{[i}{}_{[l}X^j{}_m\delta^{k]}{}_{n]} - \frac{9}{2}(v\widetilde{v})^2. \tag{6.5}$$

Comparing with the bracket $[B^{ijk}, \widetilde{B}_{lmn}]$ calculated in (4.7), we find they indeed agree if $c\widetilde{c} = -1$. Thus we conclude $c = \widetilde{c} = i$, i.e.

$$B^{ijk} = i(Y^{ijk}v), \qquad \widetilde{B}_{ijk} = i(Y_{ijk}\widetilde{v}). \tag{6.6}$$

6.2 Constraints

Now, let us check using the Joseph relations that the constraints on the SU(3) side, listed in Eqs. (4.18) $\sim$ (4.27), can be correctly reproduced on the exceptional side.

- (4.18): Contract v or $\widetilde{v}$ to the relation 2, (5.22).
- (4.19): Contract v or $\widetilde{v}$ to the relation 3, (5.23).
- (4.20): Contract v or $\widetilde{v}$ to the relation 4, (5.24).
- (4.21): This is exactly the relation 6, (5.26).
- (4.22): This is exactly the relation 7', (5.28).
- (4.23): Contract $v_\alpha v_\beta$ or $\widetilde{v}_\alpha \widetilde{v}_\beta$ to the relation 1 (5.21).

As for the relation of the type $B\widetilde{B} = MMM$,

- (4.24): The singlet part. Contract $v_\alpha \widetilde{v}_\beta$ to the relation 7 (5.27).
- (4.25): The adjoint part. Contract $v_\alpha \widetilde{v}_\beta$ to the relation 1 (5.21).
- (4.26): The $(0,1,0,1,0)$ part. Contract $v_\alpha \widetilde{v}_\beta$ to the relation 5 (5.25).
- (4.27): This is the $(0,0,2,0,0)$ part and is slightly trickier, but it follows from a cubic Joseph identity

$$0 = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} Z_{\alpha\beta} Y^{ijk}_{\gamma} Y_{lmn,\delta} \,\big|_{0,0,2,0,0} - 6 X^{[i}{}_l X^{j}{}_m X^{k]}{}_n \,\big|_{0,0,2,0,0} \tag{6.7}$$

  upon replacing $Z_{\alpha\beta}$ with $v_{(\alpha}\widetilde{v}_{\beta)}$. This cubic Joseph identity itself can be derived from the quadratic Joseph identities, as it should be. First, we use the relation 4 (5.24) to show

$$\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} Z_{\alpha\beta} Y^{ijk}_{\gamma} Y_{lmn\,\delta} \,\big|_{0,0,2,0,0} \propto X^{[i}{}_p Y^{jk]p}_{\alpha} Y_{lmn\,\beta}\epsilon^{\alpha\beta}. \tag{6.8}$$

  Now, the antisymmetric product of two Y's contain both the singlet and the $(0,1,0,1,0)$ part. One sees the singlet drops out inside the projector to the $(0,0,2,0,0)$ part, so we have

$$\propto \left(X^{[i}{}_p (Y^{jk]p} Y_{lmn}) \,\big|_{0,1,0,1,0} \right) \big|_{0,0,2,0,0}\,. \tag{6.9}$$

Then we use the relation 5 (5.25) to transform this to

$$\propto X^i{}_l X^j{}_m X^k{}_n \Big|_{0,0,2,0,0} . \tag{6.10}$$

The proportionality constant can be fixed, e.g. by evaluating on a few points on the orbit. This concludes the comparison of the constraints.

7. Conclusions

In the previous three sections, we determined the Higgs branches both on the SU(3) side and on the exceptional side. We demonstrated that their defining equations agree, and furthermore exhibited that the Poisson bracket and the conjugation are the same on both sides. As was stated in Sec. 2 and will be detailed in Appendix B, these are (almost) sufficient to conclude that they are the same as hyperkähler manifolds. To remove any remaining doubts, we compare the Kähler potentials of the two sides in Appendix C. Again, they show remarkable agreement with one another.

Thus we definitely showed the agreement of the Higgs branches of the new S-duality pair proposed by Argyres and Seiberg in [1], which provides a convincing check of their conjecture. In this paper we only dealt with the example involving E_6, but there are more examples of similar dualities in [1] and [4]. It would be interesting to carry out the same analysis of the Higgs branches to those examples. A pressing issue is to understand the Argyres-Seiberg duality more fully. For example, it would be nicer to have an embedding of this duality in string/M-theory. We hope to revisit these problems in the future.

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[4] It can be downloaded from `http://www-math.univ-poitiers.fr/~maavl/LiE/`.

A. Conventions

Greek indices α, β are for the doublets of SU(2), $a, b, c, \ldots$ for the triplets of SU(3) and $i, j, k, \ldots$ for the sextets of SU(6). We define

$$(uw) \equiv u_\alpha w_\beta \epsilon^{\alpha\beta} \tag{A.1}$$

for two doublets u_α and w_α,

We use the following sign conventions for the epsilon tensors of SU(2), SU(3) and SU(6):

$$\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}, \quad \epsilon^{abc} = \epsilon_{abc}, \quad \epsilon^{ijklmn} = \epsilon_{ijklmn}. \tag{A.2}$$

We normalize the antisymmetrizer $[abc...]$ and the symmetrizer $(abc...)$ so that they are projectors, i.e.

$$T^{ijk} = T^{[ijk]} \tag{A.3}$$

for the antisymmetric tensors T^{ijk}, etc. We raise and lower three antisymmetrized indices of SU(6) via the following rule:

$$T^{ijk} = \frac{1}{6}\epsilon^{ijklmn} T_{lmn}, \qquad T_{lmn} = \frac{1}{6} T^{ijk} \epsilon_{ijklmn}. \tag{A.4}$$

Our convention for the placement of the indices of the complex conjugate is e.g.

$$\overline{Z}^{\alpha\beta} \equiv (Z_{\alpha\beta})^*, \tag{A.5}$$

i.e. the complex conjugation is always accompanied by the exchange of subscripts and superscripts, as is suitable for the action of SU groups.

We take the Kähler potential of a flat $\mathbb{C}$ parameterized by z with the standard metric to be

$$K = |z|^2. \tag{A.6}$$

B. Twistor spaces of hyperkähler cones

Recall that a hyperkähler manifold $\mathcal{M}$ admits a continuous family of complex structures J_ζ, parameterized by $\zeta \in \mathbb{CP}^1$. The full information in the hyperkähler metric is captured by this family of complex structures and their Poisson brackets. It can be encoded into purely holomorphic data on a complex manifold $\mathcal{Z}$, the *twistor space* of $\mathcal{M}$, as we now review.

Topologically $\mathcal{Z} = \mathcal{M} \times \mathbb{CP}^1$. Its complex structure can be specified by specifying which functions on $\mathcal{Z}$ are holomorphic: they are $f(x, \zeta)$ which are holomorphic in ζ for fixed $x \in \mathcal{M}$, and also holomorphic in x with respect to complex structure J_ζ for fixed

ζ. Hence we may view $\mathcal{Z}$ as a holomorphic fiber bundle over $\mathbb{CP}^1$, where the fiber over ζ is just a copy of $\mathcal{M}$, equipped with complex structure J_ζ.

The Poisson brackets on the holomorphic functions in each fiber glue together globally to give a bracket operation on $\mathcal{Z}$. This bracket operation is globally twisted by the line bundle $\mathcal{O}(-2)$: i.e. given local holomorphic functions f_1, f_2 we get a local section $\{f_1, f_2\}$ of $\mathcal{O}(-2)$, and more generally if f_1, f_2 are sections of $\mathcal{O}(d_1)$, $\mathcal{O}(d_2)$ then $\{f_1, f_2\}$ is a section of $\mathcal{O}(d_1 + d_2 - 2)$. Finally there is an involution σ on $\mathcal{Z}$, simply defined by $(x, \zeta) \to (x, -1/\overline{\zeta})$. This is an antiholomorphic involution, since the complex structure J_ζ is opposite to $J_{-1/\overline{\zeta}}$.

As a complex manifold $\mathcal{Z}$ is a fibration over $\mathbb{CP}^1$, and (x, ζ) with x fixed gives a holomorphic section of this fibration, which is invariant under σ. The normal bundle to this section is isomorphic to the line bundle $\mathcal{O}(1)^{\oplus n}$ where n is the complex dimension of $\mathcal{M}$. Conversely, a holomorphic section of $\mathcal{Z}$ which is invariant under σ and whose normal bundle is isomorphic to $\mathcal{O}(1)^{\oplus n}$ is called a twistor line. Therefore, the points on $\mathcal{M}$ give rise to a n-dimensional family of twistor lines on $\mathcal{Z}$.

It was shown in [9] that given $\mathcal{Z}$, together with its Poisson brackets and antiholomorphic involution, one can canonically reconstruct a hyperkähler metric on the space of twistor lines. Therefore, to check that our two hyperkähler cones are the same is essentially to check that their twistor spaces $\mathcal{Z}$ are the same.

Now, the twistor space of a hyperkähler cone can be constructed from the data we described in Sec. 2, i.e. the Poisson bracket, the dilatation and the conjugation on $\mathcal{M}$. We pick one complex structure induced from the hyperkähler structure, and regard $\mathcal{M}$ as a complex manifold. We then form $\mathcal{Z}$ as a complex manifold as

$$\mathcal{Z} = ((\mathbb{C}^2 \setminus (0,0)) \times \mathcal{M})/\mathbb{C}^\times \tag{B.1}$$

where $\mathbb{C}^\times$ acts on the first factor by multiplication, and on the second factor as the natural complexification of the action of the dilatation. Then the Poisson bracket on $\mathcal{M}$ naturally induces one on $\mathcal{Z}$. We define σ on $\mathcal{Z}$ to send $(z, w, x) \in \mathbb{C}^2 \times \mathcal{M}$ to $(-\overline{w}, \overline{z}, \sigma(x))$. Then it is straightforward to check that this $\mathcal{Z}$ is the twistor space of $\mathcal{M}$, using the $\mathrm{SU}(2)_R$ action on $\mathcal{M}$ rotating three complex structures.

There is a subtle problem remaining, however. Namely, the theorem in [9] asserts that there is a component of the space of the twistor lines of $\mathcal{Z}$ which agrees metrically with the original hyperkähler manifold $\mathcal{M}$, but does not exclude the possibility that the space of twistor lines has many components, each of which is a hyperkähler manifold with the same complex structure but with a different metric. Mathematicians the authors consulted know no concrete example where this latter possibility is realized, so the authors think it quite unlikely that our two hyperkähler manifolds are the same as holomorphic symplectic manifolds but not as hyperkähler manifolds. To dispel this

last possibility, in the next Appendix we directly compare the Kähler potential of our two hyperkähler manifolds.

C. Comparison of the Kähler potential

In this Appendix, we describe the method to calculate and compare the Kähler potential of the Higgs branches on the two sides of the duality.

C.1 Exceptional side

The invariant norm of E_6 in our notation is

$$Z_{\alpha\beta}\overline{Z}^{\alpha\beta} + \frac{1}{6}Y_\alpha^{ijk}\overline{Y}^\alpha_{ijk} + X^i{}_j\overline{X}^j{}_i. \tag{C.1}$$

Therefore the correctly normalized Kähler potential is

$$K_{E_6} = 2\sqrt{Z_{\alpha\beta}\overline{Z}^{\alpha\beta} + \frac{1}{6}Y_\alpha^{ijk}\overline{Y}^\alpha_{ijk} + X^i{}_j\overline{X}^j{}_i}, \tag{C.2}$$

and the D-term for the $\mathrm{SU}(2) \subset E_6$ is

$$D^{(E_6)}_{\alpha\beta} = \frac{2}{K_{E6}}\left[Z_{\alpha\gamma}\overline{Z}^{\gamma\delta}\epsilon_{\delta\beta} + Z_{\beta\gamma}\overline{Z}^{\gamma\delta}\epsilon_{\delta\alpha} + \frac{1}{12}(Y_\alpha^{ijk}\overline{Y}^\gamma_{ijk}\epsilon_{\gamma\beta} + Y_\beta^{ijk}\overline{Y}^\gamma_{ijk}\epsilon_{\gamma\alpha})\right]. \tag{C.3}$$

We also have quarks v_α, $\widetilde{v}_\alpha$ which have

$$K_{v,\widetilde{v}} = \sum_\alpha(|v_\alpha|^2 + |\widetilde{v}_\alpha|^2) \tag{C.4}$$

and

$$D^{(v,\widetilde{v})}_{\alpha\beta} = \frac{1}{2}(v_\alpha\overline{v}^\gamma\epsilon_{\gamma\beta} + v_\beta\overline{v}^\gamma\epsilon_{\gamma\alpha} + \widetilde{v}_\alpha\overline{\widetilde{v}}^\gamma\epsilon_{\gamma\beta} + \widetilde{v}_\beta\overline{\widetilde{v}}^\gamma\epsilon_{\gamma\alpha}). \tag{C.5}$$

The Kähler potential of the exceptional side is thus given by

$$K_{v,\widetilde{v}} + K_{E_6} \tag{C.6}$$

restricted to the locus

$$v_{(\alpha}\widetilde{v}_{\beta)} + Z_{\alpha\beta} = 0, \qquad D^{(v,\widetilde{v})}_{\alpha\beta} + D^{(E_6)}_{\alpha\beta} = 0 \tag{C.7}$$

expressed as a function of M^i_j, B^{ijk}, $\widetilde{B}^{ijk}$ and their complex conjugates.

C.2 SU(3) side

We start from the Kähler potential

$$K = \sum_{i,a} |Q^i_a|^2 + |\widetilde{Q}^a_i|^2. \tag{C.8}$$

Using the analysis in [11], the Kähler potential on the quotient was determined in [16] as

$$K = 2 \sum_{i=1,2,3} \sqrt{m_i^2 + \frac{\nu^2}{4}} \tag{C.9}$$

where $(m_1^2, m_2^2, m_3^2, 0, 0, 0)$ are the eigenvalues of $M^i{}_j \overline{M}^j{}_k$, and ν is defined by

$$3\nu = \sum_{i,a} |Q^i_a|^2 - |\widetilde{Q}^a_i|^2, \tag{C.10}$$

i.e. 1/3 of the U(1) D-term. In terms of gauge invariants we have

$$\prod_{i=1,2,3} \left(\sqrt{m_i^2 + \frac{\nu^2}{4}} + \frac{\nu}{2} \right) = \frac{1}{6} B^{ijk} \overline{B}_{ijk}, \tag{C.11}$$

$$\prod_{i=1,2,3} \left(\sqrt{m_i^2 + \frac{\nu^2}{4}} - \frac{\nu}{2} \right) = \frac{1}{6} \widetilde{B}^{ijk} \overline{\widetilde{B}}_{ijk}. \tag{C.12}$$

C.3 Comparison

Now, the Kähler potentials of the two sides, (C.6) and (C.9) should agree as functions of M, B and $\widetilde{B}$, but we have not been able to check that analytically. Instead, one can check it numerically on as many points on the quotient as computer time allows. The algorithm is as follows:

1. Generate a point $\mathbb{X} = (X^i{}_j, Y^{ijk}_\alpha, Z_{\alpha\beta})$ on the nilpotent orbit of E_6, by applying an element of the group E_6 to the point $(Z_{11}, Z_{12}, Z_{22}) = (1, 0, 0)$, $X^i{}_j = Y^{ijk}_\alpha = 0$.

2. Find $v_\alpha, \widetilde{v}_\alpha$ which satisfy

   $$v_{(\alpha} \widetilde{v}_{\beta)} + Z_{\alpha\beta} = 0. \tag{C.13}$$

   This is more or less unique up to $\mathbb{C}^\times$ action on $v, \widetilde{v}$.

3. Apply $\mathrm{SL}(2, \mathbb{C})$ action to $(v, \widetilde{v}, \mathbb{X})$ to find the solution of the D-term equation,

   $$D^{(v,\widetilde{v})}_{\alpha\beta} + D^{(E_6)}_{\alpha\beta} = 0. \tag{C.14}$$

   This is equivalent to the minimization of

   $$K_{v,\widetilde{v}}(g(v), g(\widetilde{v})) + K_{E6}(g(\mathbb{X})) \tag{C.15}$$

   where g is an $\mathrm{SL}(2, \mathbb{C})$ action.

4. Form M, B, $\widetilde{B}$ from v, $\widetilde{v}$ and $\mathbb{X}$ thus obtained, and calculate ν and m_i. At this point, two checks of the sanity of the calculation are possible. One is to see that three eigenvalues of $M\overline{M}$ are zero. Another is to see that ν determined from (C.11), (C.12) is equal to
$$\nu = \sum_\alpha |v_\alpha|^2 - |\widetilde{v}_\alpha|^2. \tag{C.16}$$
The latter fact follows from the identification of ν as 1/3 of the U(1) moment map on the quotient.

5. Evaluate the Kähler potential of the SU(3) side using (C.9) and compare it to that of the exceptional side (C.6).

We implemented the algorithm above in Mathematica, and found that the value of the Kähler potential at any points agrees on both sides of the duality to arbitrary accuracy.[5] An analytic proof of the agreement of the Kähler potential will be welcomed.

D. Mathematical Summary

Let us summarize briefly in the language of mathematics what was done in this paper. Let $M(m,n)$ be
$$M(m,n) = \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V) \qquad \text{where } V = \mathbb{C}^m, \quad W = \mathbb{C}^n \tag{D.1}$$
which is a flat hyperkähler space of quaternionic dimension mn. It has a natural triholomorphic action of $\mathrm{U}(m) \times \mathrm{U}(n)$ induced from its action on V and W. Let $N(m,n)$ be the flat hyperkähler space
$$N(m,n) = \mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{H}^n \tag{D.2}$$
of quaternionic dimension mn, which has a natural triholomorphic action of $\mathrm{SO}(m) \times \mathrm{Sp}(n)$.

One then defines a hyperkähler quotient A by
$$A_1 = M(6,3) /\!/\!/ \mathrm{SU}(3). \tag{D.3}$$
We consider another hyperkähler quotient
$$A_2 = (N(2,1) \times \mathcal{O}_{\mathrm{min}}(E_6)) /\!/\!/ \mathrm{Sp}(1) \tag{D.4}$$

[5] We thank H. Elvang for improvement of the accuracy in the calculation.

where $\mathcal{O}_{\min}(G)$ is the minimal nilpotent orbit of the group G, and the $\mathrm{Sp}(1)$ action on $\mathcal{O}_{\min}(E_6)$ is given by considering the maximally compact subgroup $\mathrm{Sp}(1)\times\mathrm{SU}(6) \subset E_6$.

One sees easily that $A_{1,2}$ are both of quaternionic dimension 10, both carry a natural triholomorphic action of $\mathrm{SU}(6) \times \mathrm{U}(1)$. Our claim is that $A_1 = A_2$ as hyperkähler cones. We demonstrated that A_1 and A_2 match as holomorphic symplectic varieties by explicitly showing that their defining equations and the holomorphic symplectic forms are the same. We also found that the twistor spaces of A_1 and A_2 are the same as complex manifolds with antiholomorphic involution, but could not show that A_1 and A_2 correspond to the same family of twistor lines. Instead we directly compared the Kähler potentials of A_1 and A_2. Again we could not rigorously prove the equivalence, but we performed numerical calculations of the Kähler potential which convinced us that they agree.

The equivalence of $A_{1,2}$ was suggested by the analysis of a new type of S-duality in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in [1]. In [1, 4], more examples of the same type of duality were described, of which we record two more here.

Now consider

$$B_1 = N(12,2)/\!\!/\!\!/\mathrm{Sp}(2) \tag{D.5}$$

and

$$B_2 = \mathcal{O}_{\min}(E_7)/\!\!/\!\!/\mathrm{Sp}(1). \tag{D.6}$$

Here $\mathrm{Sp}(1)$ acts on $\mathcal{O}_{\min}(E_7)$ through the maximal subgroup $\mathrm{Sp}(1)\times\mathrm{SO}(12) \subset E_7$. The quaternionic dimension of $B_{1,2}$ is 14, and both have triholomorphic actions of $\mathrm{SO}(12)$. We believe $B_1 = B_2$ as hyperkähler cones.

For an example which involves $\mathcal{O}_{\min}(E_8)$, consider

$$C_1 = (Z \oplus N(11,3))/\!\!/\!\!/\mathrm{Sp}(3). \tag{D.7}$$

Here Z is a pseudoreal irreducible representation of $\mathrm{Sp}(3)$ of quaternionic dimension 7, which arises as

$$\wedge^3_{\mathbb{C}} X = Z \oplus X \tag{D.8}$$

where $X = \mathbb{C}^6$ is the defining representation of $\mathrm{Sp}(3)$. Let us take another hyperkähler quotient

$$C_2 = \mathcal{O}_{\min}(E_8)/\!\!/\!\!/\mathrm{SO}(5) \tag{D.9}$$

where $\mathrm{SO}(5)$ acts via embedding

$$\mathrm{SO}(5) \times \mathrm{SO}(11) \subset \mathrm{SO}(16) \subset E_8. \tag{D.10}$$

It is easy to check that $C_{1,2}$ are both of quaternionic dimension 19, and $\mathrm{SO}(11)$ acts triholomorphically on both C_1 and C_2. We predict that $C_1 = C_2$ as hyperkähler cones.

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